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This report contains results of studies of factorization of matrices with displacement structure: A unified treatment of matrices with strongly regular and arbitrary rank profiles; linear system stability and root distribution of polynomials: A unified approach via inertia computation of Bezoutians; fast parallelizable array algorithms via displacement representations of matrices. (KR) ←

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1. Introduction

Our research project with ARO started with a small effort in 1977 to explore certain connections we have noted between inverse-scattering theory (Gelfand-Levitan, Marchenko, etc.) and linear least-squares estimation. As indicated by our list of publications with ARO support, this has been a very fruitful area of research. We shall not summarize all our results here; we note, however, that one important result is our establishment of a fundamental correspondence between one-dimensional inverse scattering in a lossless medium and efficient triangular factorization of certain structured (close-to-Toeplitz) matrices. Then by broadening the notion of structured matrices we were able to recast many signal processing problems as (non-classical) inverse scattering problems, e.g., partial realization, decoding of error-correcting codes, and polynomial zero-location. This led us to develop a general scheme (see Lev-Ari and Kailath (1986)) that can be applied to solve each one of these generalized inverse scattering problems and, at the same time, to compute the triangular factors of a certain associated structured matrix.

Our work has led us to realize that these seemingly unrelated problems can be grouped into two distinct families:

(1) **Inverse scattering problems,** which involve the reconstruction of internal parameters of an unknown system from given boundary (i.e., input/output) data.

(2) **Matrix factorization problems,** which involve obtaining the inertia and often also the triangular (lower-diagonal-upper) factorization of a given Hermitian matrix.

We showed that there exists a direct correspondence between these two families of problems via the notion of energy conservation or *losslessness*. For this, we first introduced a procedure for associating Hermitian matrices with incident and reflected waves in a lossless cascade network and then showed that the information required to construct the triangular factorization of these matrices was provided by the internal signals in the network. On the other hand, we focused on a particular procedure for solving the inverse scattering problem, known as 'layer peeling' [Bruckstein and Kailath (1987)]. Here, the unknown system is identified by layers or sections: the outermost layer is identified first, then peeled off to reveal the input/output signals associated with the next layer. Translating this procedure to matrix language results in an extension of the well-known Cholesky factorization method: the triangular factor of a given matrix is computed recursively, column-by-column. Each time a new column has been computed its effect is removed by subtracting a suitable matrix of rank one. The recursive nature of these layer-peeling and factorization procedures results in a

cascade structure for the associated network models. Thus both families of problems are, in fact, two facets of the same phenomenon, and the same relation exists between the recursive procedures used to solve these problems.

Moreover, with an appropriate formulation we were able to extend our conceptual framework to cascade networks that display a rather generalized concept of losslessness [Lev-Ari and Kailath (1986)]. It may be useful to recall here that cascade network models abound in the circuits and systems literature. They are central to the classical filter synthesis methods of Darlington and Brune, as well as to the modern filter synthesis methods of Deprettere and Dewilde (1980), Rao and Kailath (1984), and Vaidyanathan and Mitra (1984). They arise in the stochastic filtering of stationary and nonstationary processes, as well as in the related problems of efficient factorization and inversion of Toeplitz and near-Toeplitz matrices [see, e.g., Lev-Ari and Kailath (1984), and Kailath (1986)] and extensions thereof [Lev-Ari and Kailath (1986)]. They are implicit in much of the research on fast algorithms for Toeplitz and Hankel matrices [see, e.g., Friedlander, Morf, Kailath and Ljung (1979), and Heinig and Rost (1984)]. Since the layered earth model of Gopilaud is a lossless cascade network, the inverse scattering literature is also concerned with such networks [see e.g., Robinson (1975), Bruckstein and Kailath (1987), Bruckstein, Lévy and Kailath (1985)]. Finally, such networks arise in the solution of several problems in system theory, namely in the construction of efficient tests for stability and zero location [see, e.g., Jury (1964)], in the solution of the partial realization problem [Citron, Bruckstein and Kailath (1984)] and of the stochastic realization problem [Mullis and Roberts (1986)], and recently also in model order reduction techniques [Ball and Gohberg (1986)].

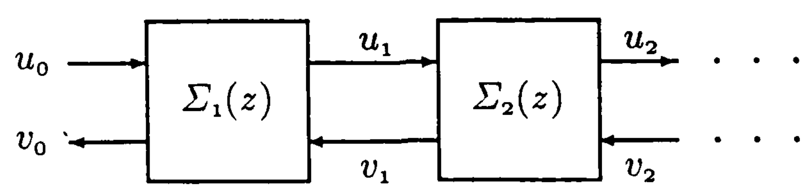

Fig. 1 Lossless Cascade Network

It will also be helpful to be more specific about the connections between cascade networks, inverse-scattering and matrix factorization. Consider a cascade of sections (or layers), with p discrete-time *inbound signals* $\mathbf{u}$, flowing from the input terminals into the system, and q *output signals* $\mathbf{v}$ flowing out of the system to the output terminals (Fig. 1). This cascade system is called (internally) lossless if the energy (= l_2 norm) input into each section equals the energy output from the same section, viz.,

$$\|\mathbf{u}_{i-1}\|^2 + \|\mathbf{v}_i\|^2 = \|\mathbf{u}_i\|^2 + \|\mathbf{v}_{i-1}\|^2 \quad , \quad i \geq 1 \tag{1.1}$$

where $\mathbf{u}_i$, $\mathbf{v}_i$ denote the signals flowing through the boundary between the i-th and the $(i+1)$-th layer. Alternatively, we may characterize losslessness in terms of the *scattering matrices* $\Sigma_i(z)$, which relate the Z-transforms of these signals, i.e.,

$$\begin{bmatrix} u_i(z) \\ v_{i-1}(z) \end{bmatrix} = \Sigma_i(z) \begin{bmatrix} u_{i-1}(z) \\ v_i(z) \end{bmatrix} \tag{1.2}$$

It is well-known that a necessary and sufficient condition for losslessness is

$$\Sigma_i(z)[\Sigma_i(1/z^*)]^* = I \tag{1.3}$$

where the asterisk (*) denotes Hermitian transpose (complex conjugate for scalars). Often we need to use the *chain-scattering matrices* $\Theta_i(z)$, which relate the signals at one side of each layer to the signals on the other side, i.e.,

$$\begin{bmatrix} u_i(z) \\ v_i(z) \end{bmatrix} = \Theta_i(z) \begin{bmatrix} u_{i-1}(z) \\ v_{i-1}(z) \end{bmatrix} \quad , \quad i \geq 1 \tag{1.4}$$

Since the losslessness constraint can be rewritten in the form $\|\mathbf{u}_i\|^2 - \|\mathbf{v}_i\|^2 = \|\mathbf{u}_{i-1}\|^2 - \|\mathbf{v}_{i-1}\|^2$ it follows that the chain-scattering matrices are *J-orthogonal,* i.e.,

$$\Theta_i(z) J [\Theta_i(1/z^*)]^* = J \quad , \quad J := diag\{I_p , -I_q\} \tag{1.5}$$

The layer peeling procedure can now be described as follows:

> Beginning with the given boundary data $u_0(z)$ and $v_0(z)$ repeat the following steps for $i = 1, 2, \ldots$
>
> (a) Identify the internal parameters of the i-th section from $u_{i-1}(z)$, $v_{i-1}(z)$.
>
> (b) Compute $u_i(z)$, $v_i(z)$ via (1.4).

Notice that the losslessness property is not required in order to carry out this procedure, only that we are able to identify each section from its input/output data.

This observation, first made by Bruckstein and Kailath (1987), indicates that the layer-peeling procedure may apply to cascade networks that do not satisfy the standard notion of losslessness (1.3).

Losslessness is, nevertheless, essential to the association of inverse scattering problems with matrix factorization problems, as demonstrated by Kailath, Bruckstein and Morgan (1986). They analyze the layer-peeling procedure associated with a *lossless transmission-line model*, where

$$\Theta_i(z) = \frac{1}{\sqrt{1 - |k_i|^2}} \begin{bmatrix} 1 & -k_i \\ -k_i^* & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \tag{1.6}$$

is a chain-scattering matrix with a single storage element (z). They show that the principle of energy conservation results in an identity of the form

$$\mathbf{L}(u_0)\mathbf{L}^*(u_0) - \mathbf{L}(v_0)\mathbf{L}^*(v_0) = \mathbf{X}\mathbf{X}^* \tag{1.7}$$

where $\mathbf{X}$ is a lower-triangular matrix whose j-th column consists of the samples of the signal stored in the j-th section of the network, and $\mathbf{L}(u_0)$ (resp. $\mathbf{L}(v_0)$) is a lower-triangular Toeplitz matrix constructed from the coefficients of the Z-transform $u_0(z)$ (resp. $v_0(z)$), namely

$$\mathbf{L}(f) := \begin{bmatrix} f_0 & & & \\ f_1 & f_0 & & O \\ f_2 & f_1 & f_0 & \\ \cdot & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & & \cdot \end{bmatrix} \tag{1.8a}$$

where

$$f(z) := \sum_{i=0}^{\infty} f_i z^i \tag{1.8b}$$

is a Z-transform of a signal $\mathbf{f} = \{f_i \ ; \ 0 \le i < \infty\}$. Thus, the Hermitian matrix

$$\mathbf{R} := \mathbf{L}(u_0)\mathbf{L}^*(u_0) - \mathbf{L}(v_0)\mathbf{L}^*(v_0) \tag{1.9}$$

can be factored by applying an appropriate layer-peeling procedure to the pair of signals $\{u_0(z), v_0(z)\}$ and reading off the elements of the triangular factor $\mathbf{X}$ from the signals stored in the delay element in each section.

The association between inverse-scattering problems and efficient matrix factorization has motivated us to look for a similar association involving structured matrices that are not of the form (1.9), e.g., Hankel matrices, sums of Toeplitz and

Hankel matrices, Bezoutians with respect to various curves, and Vandermonde matrices. The key to the extension of the results (1.7)-(1.9) beyond the standard notion of losslessness is the observation that the matrix $\mathbf{R}$ in (1.9) has a very specific *displacement structure* [see, e.g., Lev-Ari and Kailath (1984), and Kailath (1986)]. One convenient way to describe this structure is via the bivariate Z-transform (= generating function) of $\mathbf{R}$, viz.,

$$R(z,w) := \left[\, 1 \;\; z \;\; z^2 \ldots \right] \mathbf{R} \left[\, 1 \;\; w \;\; w^2 \ldots \right]^* \tag{1.10}$$

Applying this definition to (1.9) we obtain

$$R(z,w) = \frac{u_0(z)u_0^*(w) - v_0(z)v_0^*(w)}{1 - zw^*} \tag{1.11}$$

which, for various choices of $\{u_0(z),\, v_0(z)\}$ describes the so-called quasi-Toeplitz family of matrices [see Lev-Ari and Kailath (1986)]. The quasi-Hankel family of matrices, which arises in the study of Bezoutians, also has a simple form, viz.,

$$R(z,w) = \frac{a(z)b^*(w) - b(z)a^*(w)}{z - w^*} \tag{1.12}$$

A comparison of (1.11) with (1.12), as well as our previous work in this area suggests a generalization to matrices whose generating functions are of the form

$$R(z,w) = \frac{G(z)\,J\,G^*(w)}{d(z,w)} \tag{1.13}$$

where $G(z)$ is a row vector of functions, J is a constant nonsingular Hermitian matrix of suitable dimensions and the *displacement function* $d(z,w)$ is a fixed function with the Hermitian property $d^*(z,w) = d(w,z)$. The triplet $\{d(z,w),\, G(z),\, J\}$ has been called a *generator* of the Hermitian matrix $\mathbf{R}$, since it uniquely determines $R(z,w)$ (and hence also $\mathbf{R}$) via (1.13) [for more details, see Lev-Ari and Kailath (1986)].

In Section 2 we review the generalized factorization algorithm we developed for structured matrices with generating functions of the form (1.13). However, these results were developed for strongly-regular matrices, viz., those with non-vanishing leading singular minors. Such singularities have not been easy to handle, even in the special cases of Toeplitz and Hankel matrices, though much more progress has been made for Hankel matrices. Our generating function approach seems to allow the development of efficient procedures, which involve only scalar computations, for overcoming singularities in the factorization. In contrast, the occurrence of singularities in non-structured matrices can be overcome only by inverting a matrix

whose size is proportional to the depth of the singularity.

In another direction, we have noticed that our results can be applied to the much-studied (for nearly 150 years) problems of determining the stability or, more generally, the root-distribution with respect to a given curve in the complex plane (e.g., the imaginary axis or the unit circle). This is because by a classical result of Hermite (1856), the root- distribution is determined by the inertia of a certain structured matrix (known as Bezoutian), which has displacement structure in our sense (see (1.11)-(1.13)). It turns out that our general factorization scheme readily yields the classical Routh-Hurwitz procedure for the imaginary axis and the classical Schur-Cohn procedure for the unit circle [see Lev-Ari, Bistritz and Kailath (1987)]. Moreover, as will be described in more detail in Section 3, our approach shows that there exist many alternatives to these classical procedures, some of which are more efficient than the classical ones. Our work provides an organized and complete approach to this much-studied subject, relating old tests to each other and indicating where new tests remain to be developed and explored.

Finally, in Section 4 we describe yet another direction of research on displacement structure that leads to several new results, especially on orthogonal (QR) rather than triangular (LDU) factorization and correspondingly on least-squares rather than exact solutions of linear equations. The main point here is to go back to the matrix form of the generating function representation (1.13), by 'inverting' the Z-transform operation (1.10). For instance, the resulting matrix equivalent of (1.11) is

$$\mathbf{R} - \mathbf{Z}\,\mathbf{R}\,\mathbf{Z}^* = \mathbf{u}_0\mathbf{u}_0^* - \mathbf{v}_0\mathbf{v}_0^* \tag{1.14}$$

where $\mathbf{Z}$ is the lower shift matrix with 1's on the first subdiagonal and 0's elsewhere, and $\mathbf{u}_0$, $\mathbf{v}_0$ are column vectors consisting of the coefficients of the power series expansions of $\mathbf{u}_0$, $\mathbf{v}_0$, respectively. It turns out that the main features of our structured factorization procedure are preserved even when we replace $\mathbf{Z}$ and $\mathbf{Z}^*$ in (1.14) by arbitrary (lower-triangular) matrices $\mathbf{F}_1$ and $\mathbf{F}_2$, so that

$$\mathbf{R} - \mathbf{F}_1\,\mathbf{R}\,\mathbf{F}_2^* = \mathbf{u}\mathbf{u}^* - \mathbf{v}\mathbf{v}^* \tag{1.15}$$

A matrix $\mathbf{R}$ that satisfies this equation admits a structured factorization procedure even though (1.15) does not correspond any more to a simple generating function representation like (1.13). In fact, a general (efficient) factorization scheme can be constructed for all matrices $\mathbf{R}$ that satisfy a displacement equation of the form

$$\mathbf{F}_1\,\mathbf{R}\,\mathbf{F}_2^* - \mathbf{F}_3\,\mathbf{R}\mathbf{F}_4^* = \mathbf{u}\mathbf{u}^* - \mathbf{v}\mathbf{v}^* \tag{1.16}$$

where $\mathbf{F}_i$ for $i=1,2,3,4$ are arbitrary lower-triangular matrices. Furthermore, the

same general scheme can be used to obtain efficient algorithms for matrix inversion (exact and least-squares) and QR factorization; this is achieved by applying the general factorization procedure to a suitable *composite matrix*, which is constructed from the given matrix **R**. We have recently begun to explore the application of such composite matrix formulations to new problems, as described in Section 4.

2: Factorization of Matrices with Displacement Structure: A unified treatment of matrices with strongly regular and arbitrary rank profiles.

In this section we first review in some detail our generalized scheme for strongly-nonsingular structured matrices, and apply it for illustration, to Hankel and Quasi-Hankel matrices. Then in Section 2.2, we examine the problems caused by the presence of zero minors.

2.1: Factorization of Strongly Nonsingular Matrices with Generalized Displacement Structure.

In 1986, Lev-Ari and Kailath [1986] extended this earlier work on Toeplitz oriented structures to a much broader family of structured matrices, including Hankel matrices and their inverses, Bezout matrices, sums of Hankel and Toeplitz matrices and several other interesting matrices. Motivated by Bistritz's [1984] stability test for discrete time system (which is a 3-term immittance recursion for inertia computation of the related Bezout matrix) Bistritz, Lev-Ari and Kailath [1987] developed the so-called 3 term immittance formulation of Levinson and Schur algorithms for Hermitian Toeplitz and Quasi Toeplitz (congruent to Toeplitz) matrices. Recently Bistritz and Kailath [1988] developed extensions of Schur and Levinson algorithms to non-symmetric Toeplitz and non-symmetric Quasi-Toeplitz matrices. The concept of generating functions was the starting point for much of this analysis and we shall start at that point here. We should mention that all the results need the assumption that all the leading minors of the associated matrices have to be nonzero - this is the assumption of ''strong nonsingularity''.

The key idea behind all these fast algorithms is the concept of displacement rank (see Kailath, Kung and Morf [1979]), as extended by Lev-Ari and Kailath [1986]. The concept of displacement structure and its properties can be appropriately described by their generating functions. The generating function of a matrix R is the following bivariate form, viz.

$$R(z,w) = [1 \; z \; z^2 \; .... \,] \, R \, [1 \; w \; w^2 \; .... \,]^* \tag{2.1}$$

The generating function of a Hermitian matrix with generalized displacement structure (Lev-Ari and Kailath [1986]) assumes the following form,

$$R(z,w) = \frac{G(z)J\,G^*(w)}{d(z,w)} \tag{2.2}$$

where J is any constant non singular Hermitian matrix, $d(z,w)$ is a Hermitian generating function like $R(z,w)$, viz.

$$d(z,w) = [1\ z\ z^2\ ....]\ J_d\ [1\ w\ w^2\ ....]^* \tag{2.3}$$

where J_d is a constant (possibly singular) Hermitian matrix. As will be seen below, common choices are $d(z,w) = 1-zw^*$ (for Toeplitz-related matrices) and $d(z,w) = z-w^*$ (for Hankel-related matrices). It has been shown in Lev-Ari and Kailath [1986] that fast factorization of R is possible if there exist matrix functions $\Theta_i(z)$ and constant matrices $J_i (J_0 = J)$ such that

$$\Theta_i(z)\, J_{i+1} \Theta_i^*(w) = J_i - \frac{d(z,w)}{d(z,\xi_i) d(\xi_i,w)} J_i M_i J_i \tag{2.4}$$

where

$$M_i = G_i^*(\xi_i)\, R_i^{-1}(\xi_i,\xi_i)\, G_i(\xi_i) \ , \tag{2.5}$$

$$R_i(z,w) = \frac{G_i(z)\, J_i G_i^*(w)}{d(z,w)} \ ; \ G_0(z) = G(z) \ , \tag{2.6}$$

and ξ_i is arbitrary.

They showed that such $\{\Theta_i\}$ exist if and only if the $\{J_i\}$ are congruent to J and $d(z,w)$ has the form

$$d(z,w) = \gamma(z)\gamma^*(w) - \delta(z)\delta^*(w) \ . \tag{2.7}$$

Fast factorization of R is obtained via the recursion

$$(z-\xi_i)\, G_{i+1}(z) = G_i(z)\, \Theta_i(z) \tag{2.8}$$

where

$$\Theta_i(z) = \left\{ I - \frac{d(z,\tau_i)}{d(z,\xi_i) d(\xi_i,\tau_i)} J_i M_i \right\} U_i \ . \tag{2.9}$$

for U_i matrices that satisfy

$$U_i J_{i+1} U_i^* = J_i \tag{2.10}$$

and τ_i any complex constant such that $d(\tau_i,\tau_i) = 0$.

The choice $\xi_i = 0$ in (2.8) results in triangular factorization. In fact (for $\xi_i = 0$)the quantity $R_i(z,w)$ in equation (2.6) is the generating function of the Schur complement of the top left entry of the matrix R_{i-1}. Other choices of ξ_i result in non triangular factorizations. It is crucial for the existence of any fast triangular factorization procedure that the Schur complement R_{i+i} inherits the displacement structure of R_i. Equations (2.4) - (2.6) represents a slightly generalized version of this fact.

If the triangular factorization of R is

$$R = LDL^* \tag{2.11}$$

where $L = [\mathbf{l}_0,\mathbf{l}_1,...,\mathbf{l}_{n-1}]$, lower triangular with unit diagonal and $D = diag\{d_i\}$, then we can recursively compute the factorization (2.11) by the formula (2.8) remembering that

$$d_i = R_i(0,0) \text{ and } \mathbf{l}_i(z) = z^i R_i(z,0)/d_i \tag{2.12}$$

for

$$\mathbf{l}_i(z) = [\ 1\ z\ z^2\ ....]\ \mathbf{l}_i \ . \tag{2.13}$$

Example: Hankel Matrices

Let us demonstrate the method by considering the family of Hankel matrices. As mentioned earlier, Hankel matrices do possess a generalized displacement structure. A Hankel matrix of order n (= 1, 2, , ...) is the name of a matrix of the form $H_{n-1}=[\ h_{i+j}\ ]_{i,j=0}^{n-1}$, where $\{\ h_k\ \}$ s are arbitrary (k=0,1, $\cdots$,2n−2). One write more explicitly:

$$H_{n-1} = \begin{bmatrix} h_0 & h_1 & h_2 & . & . & h_{n-2} & h_{n-1} \\ h_1 & h_2 & h_3 & . & . & h_{n-1} & h_n \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ h_{n-2} & h_{n-1} & h_n & . & . & h_{2n-4} & h_{2n-3} \\ h_{n-1} & h_n & h_{n+1} & . & . & h_{2n-3} & h_{2n-2} \end{bmatrix} .$$

Since triangular factorization is nested, factorization of the leading principal submatrices does not depend on the actual size of the matrix. Hence we can consider without loss of generality that the matrix under consideration is in fact semi-infinite, i.e. $0 \le i,j \le \infty$. We define its semi-infinite extension $H_{n-1,\infty}$ such that the associated generating function $H(z,w)$ is given by,

$$H(z,w) = \frac{zh(z)-w^* h^*(w)}{z-w^*} \ , \tag{2.14}$$

where

$$h(z) = \sum_{k=0}^{2n-2} h_k z^k \tag{2.15}$$

Hence $H(z,w)$ can be represented in the form (2.2) for

$$G(z) = [\ 1 \quad zh(z)\ ]\ ,\ J = antidiag[-j,j] \text{ and } d(z,w) = j(z-w^*)\ . \tag{2.16}$$

It is clear that $d(z,w)$ can be expressed as $\gamma(z)\gamma^*(w) - \delta(z)\delta^*(w)$ as in (2.7), where

$$\gamma(z) = \frac{1+jz}{\sqrt{2}} \quad \text{and} \quad \delta(z) = \frac{1-jz}{\sqrt{2}} \tag{2.17}$$

Thus by choosing $\zeta = 0$ (necessary for triangular factorization) and $\tau = \infty$ (since $d(z,z)=0$ represents the real line here),

$$G_1(z) = G(z)\Theta_0(z) \ , \tag{2.18}$$

where

$$\Theta_0(z) = \begin{bmatrix} 1 & 0 \\ z^{-1}k_0 & 1 \end{bmatrix} U_0 \ , \tag{2.19}$$

where $k_0 = h_0$.

Therefore the generating function of the Schur complement $H_1(z,w)$ of h_0 is,

$$H_1(z,w) = \frac{h_1(z)h^*(w) - h_1^*(w)h(z)}{z - w^*} \ , \tag{2.20}$$

where $h_1(z) = \sum_{k=0}^{2n-3} \frac{h_{k+1}}{h_0} z^k$. The diagonal element $d_0 = h_0$ and the generating function $l_0(z)$ of the first column of the triangular factor of $H_{n-1,\infty}$ is $\frac{h(z)}{h_0}$. Therefore the first column l_0 of the triangular factor L of H_{n-1} is $[1, \frac{h_1}{h_0}, \ldots, \frac{h_{n-1}}{h_0}]$. It is clear from (2.20) that the Schur complement of the top left entry of a Hankel matrix is not Hankel although the displacement structure of both is the same. Matrices with this structure are the so called Quasi-Hankel matrices (see Lev-Ari and Kailath [1986]). A matrix Q with the generating function

$$Q(z,w) = \frac{f(z)g^*(w) - f^*(w)g(z)}{z - w^*} \tag{2.21}$$

where $f(.)$ and $g(.)$ are polynomials in z is defined to be Quasi-Hankel. It can be easily shown that the Schur complement of the top left entry of a Quasi-Hankel matrix is also Quasi-Hankel. Thus developing recursions for factorization of Quasi-Hankel matrices is really the problem at hand. The triangular factorization of a Quasi-Hankel matrix Q with the generating function (2.21) can be carried out by the recursion (see Lev-Ari and Kailath [1986])

$$zG_{i+1}(z) = G_i(z)\Theta_i(z) \ ; \ G_0(z) = [\, f(z) \, , g(z) \,] \tag{2.22}$$

where

$$\Theta_0(z) = \begin{bmatrix} \alpha_i^{-1} & \beta_i \\ 0 & \alpha_i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z^{-1}k_i & 1 \end{bmatrix}, \tag{2.23}$$

where $k_i = sgn\,[Q_i(0,0)]$ and ,

$$Q_i(z,w) = \frac{f_i(z)g_i^*(w) - f_i^*(w)g_i(z)}{z - w^*}\ . \tag{2.24}$$

The coefficients α_i and β_i are obtained from

$$[\ \alpha_i \quad \beta_i\ ] = [k_i Q_i(0,0)]^{-1/2} G_i(0)\ . \tag{2.25}$$

The formulation (2.4) - (2.8) allows great flexibility in the selection of the scalars ξ_i, τ_i and the matrices U_i . For the LDL^* factorization problems we can choose $\{\tau_i\}$ and $\{U_i\}$ to possibly reduce computation ($\xi_i = 0$ in this case). As a matter of fact, Bistritz, Lev-Ari and Kailath [1987] have reduced computational requirements of the classical Schur and Levinson algorithms by choosing proper U_i.

2.2: Factorization of Structured Matrices with Arbitrary Rank Profiles

All our previous work on efficient factorization of matrices with displacement structure always assumed certain regularity of the rank profile of the matrix. This amounts to assuming that either all the leading principal submatrices are full rank or if a leading principal submatrix is rank deficient then all the following leading principal submatrices are rank deficient too. However, this does not encompass all the possibilities. Full rank matrices with singular leading principal submatrices are often encountered while dealing with Hankel and Bezout matrices These matrices naturally arise in the partial realization problem and the problem of root distribution of a polynomial (see Section 3). Also the Toeplitz matrices associated with the eigen filter design problem and certain buffer design problems need not have regular rank profile. So the study of factorization and inversion of structured matrices with arbitrary rank profile is indeed an important issue and as mentioned in the introduction it has recently been attacked in different ways by several researchers (in several countries). Our approach is outlined below.

The basic recursion step (2.8) assumes that $R_i(\xi_i, \xi_i) \neq 0$. Thus for triangular factorization this becomes $R_i(0,0) = 0 \Longleftrightarrow$ leading principal minors do not vanish. The classical remedy via permutation of the entries of R_i would destroy the underlying displacement structure of R_i and hence cannot be used to obtain fast algorithms. Vaidyanathan and Mitra [1987] have suggested moving the point ξ_i such that

$R_i(\xi_i, \xi_i) = 0$ and the recursion (38) can be carried on. However it would not be suitable for predetermined values of ξ_i , as for example the commonly used choice $\xi_i = 0$. Actually the choice of ξ_i affects the computational complexity of evaluating $G_i(\xi_i)$; the best choice from this point of view is $\xi_i = 0$ or $\xi_i = \pm 1$.

So we have chosen a different path based on the fact that a Hermitian matrix with singular leading principal submatrices has a "modified triangular factorization" of the form ,

$$R = LDL^* \tag{2.26}$$

where, L is lower triangular with unity diagonals and D contains non-zero scalar as well as block entries only along the main diagonal. The sizes of these block diagonal entries can be determined from the rank profile of R .

So even if $R_i(0,0)$ is zero, it may be possible to compute $G_{i+1}(z)$ from $G_i(z)$, where t is the size of the block in D whose top left element occupies the (i,i) entry in D . If it is possible to do so then we shall be able to compute the block Schur complement of the $t \times t$ leading principal submatrix of R_i if all the $j \times j$ $(1 \le j \le t-1)$ leading principal minors of R_i vanish, directly using the information in $G_{i+1}(z)$. In fact if we can formulate the transition from $G_i(z)$ to $G_{i+1}(z)$ as a sequence of t single steps, viz

$$G_i(z) \to G_{i+1}(z) \to \ldots . \to G_{i+1}(z) \tag{2.27}$$

then the process of computing this modified triangular factorization can be kept completely recursive and would allow us to determine the triangular factor L , column by column rather than by block columns. Truly then there is a need for deriving a modified version of (2.8) so as to enable us to compute the chain (2.27)

As will be described below, we have solved this problem for Hankel and Quasi-Hankel matrices (see below) and the Bezout matrices connected with the root distribution problem w.r.t the imaginary axis. We are currently working towards similar results for Toeplitz and Bezout matrices associated with root distribution problems w.r.t. the unit circle.

Hankel and Quasi-Hankel Matrices with Arbitrary Rank Profile

It is known that Hankel matrices can be inverted by fast algorithms that require $O(n^2)$ operations [see Trench (1965), Berlekamp (1968), Massey (1969), Gohberg and Semencul (1972), Kung (1977), Heinig and Rost (1984), Gohberg Kailath and Koltracht (1985) and Citron (1986)). However, the problem of efficient factorization

of Hankel matrices is not as well studied as the inversion problem. Rissanen (1971) and more recently Kung (1977) have considered this problem in the context of partial realization problems. Very recently Lev-Ari and Kailath (1986) have considered factorization of "strongly regular" Hankel ?nd Quasi-Hankel matrices as a special case of their work.

Except for Lev-Ari and Kailath 1986) other authors did not consider the factorization problems associated with Quasi-Hankel matrices. As far as factoring Hankel, Quasi-Hankel matrices from the same framework is considered, Lev-Ari and Kailath (1986) provide the results directly or indirectly for matrices that are strongly regular. However they did not consider matrices that are not strongly regular.

Rissanen and Kung both have considered Hankel matrices only. Nevertheless they have considered matrices that need not be strongly regular. However the factorization obtained by these authors differ from the form of factorization (i.e., LDL^T where L is lower triangular with unit diagonal and D is either diagonal or block diagonal) considered in this paper. Rissanen (1971) obtains a LU factorization of the given Hankel Matrix, where L is a lower triangular matrix with unit diagonal elements and U is an "upper triangular-staircase" matrix (see Rissanen (1971) for details). U becomes a perfect upper triangular matrix when the given Hankel matrix is strongly regular. Rissanen's method is recursive but it is not a fast procedure. Kung (1977) on the other hand obtains a different $(HU = L)$ factorization of the Hankel matrix H where U is an upper-triangular matrix with unit diagonal elements and L is a lower "triangular-staircase" matrix (see Kung (1977) for more details). L becomes a perfect lower triangular matrix when the given Hankel matrix is "strongly regular". Kung's method is fast as well as recursive. However Kung's procedure is not completely recursive in the sense that it makes use of block pivots in presence of vanishing principal minors. Kung's procedure requires computation of an inner product at each step. Thus this procedure is not completely parallelizable. Determination of the number of consecutive zero principal minors for a block step is not straightforward in this procedure and it requires computation of certain "predicted Markov parameters" until a mismatch between these "predicted" and the "given" Markov parameters is observed (see Kung (1977) for details).

Recently Citron (1986) considered the problem of solving a Hankel system of equations. Citron refined Kung's method to avoid block pivots and computation of inner products. Determination of the number of consecutive zero minors for block step too has been greatly simplified by Citron in Citron (1986). However the problem of fast and recursive computation of a triangular factorization of the form LDL^T where

L is lower triangular with unit diagonal and D is either diagonal or block diagonal was not explicitly solved for a Hankel matrix (possibly with arbitrary rank profile).

Below we describe a recursive procedure which is based on the fact that the Schur complement (or the block Schur complement) of the top left entry (or the block) of a Hankel or a Quasi-Hankel matrix is Quasi-Hankel. Except for Lev-Ari and Kailath (1986) the other earlier papers did not make use of this fact. Our result generalizes their results by removing the requirement of "strong regularity".

Our procedure does not require computation of inner products and it is completely parallelizable. Determination of the size of a block factor is trivially done by counting the number of repeated zeros at the origin of a polynomial (just as in Citron(1986)).

Since our approach explicitly computes D without increasing the operation count it is possible to utilize our method to compute inertia of these matrices with no extra effort. Over and above since the block diagonal entries of D are either lower triangular Hankel matrix it is possible to compute their inertia by Iohvidov's (1982) rules.

The recursions (2.21)-(2.25) for triangular factorization of the Quasi-Hankel matrices can't be used if $Q_i(0,0)$ is zero. There are three different cases that characterize a singularity (i.e. a situation with $Q_i(0,0)=0$), viz.

$$(i) \quad Q_i(z,w) \equiv 0 \ . \tag{2.28}$$

$$(ii) \quad Q_i(z,0) = 0 \ , \ Q_i(0,w) = 0 \quad \text{but} \quad Q_i(z,w) \neq 0 \ . \tag{2.29}$$

$$(iii) \quad Q_i(z,0) \neq 0 \ \text{ and } \ Q_i(0,w) \neq 0 \ .$$

The first case is trivial and indicates the end of the factorization procedure. In the second case the first row and the first column both are zeros whereas the matrix Q isn't identically zero. This situation does not pose a real problem and the recursive procedure (2.22) can be continued by replacing (2.22)-(2.23) with (see Pal and Kailath [1988] for more details)

$$z\,[\ f_{i+1}(z)\ g_{i+1}(z)\ ] = [\ f_i(z)\ g_i(z)\ ] - [\ f_i(0)\ g_i(0)] \ . \tag{2.31}$$

The corresponding $d_i = 0$ and $\mathbf{l}_i(z)$ can be arbitrarily assigned to be z^i. The third case is by far the hardest. We proceed to analyze this situation starting with the following general observation about the generating functions of Quasi-Hankel matrices.

"Given a Quasi-Hankel matrix Q it is always possible to find two polynomials $f(z)$ and $g(z)$ such that at least one of these two polynomials vanishes at the origin while satisfying (2.21)". The proof is straight forward. Therefore we can assume

without loss of generality that $f_i(0)$ is zero. Then it can be easily shown that $\lim_{z \to 0} \frac{f(z)}{z} = 0$ too. In fact, let t be the smallest positive integer such that

$$\lim_{z \to 0} z^{-t} f(z) \neq 0 .$$

Then it turns out that the first nonsingular principal submatrix of Q_i is of dimension t. The key result then is that the generating function $Q^c{}_{i,t}(z,w)$ of the block Schur complement of this $t \times t$ block is

$$Q^c{}_{i,t}(z,w) = \frac{f_{i+t}(z) g^*_{i+t}(w) - g_{i+t}(z) f^*_{i+t}(w)}{z - w^*} \tag{2.32}$$

where $f_{i+t}(z)$ and $g_{i+t}(z)$ can be computed by the recursion

$$f_{i+t}(z) = z^{-t} f_i(z) \ , \tag{2.33}$$

$$z g_{j+1}(z) = g_j(z) - k_j f_{i+t}(z) \ ; \ k_j = \frac{g_j(0)}{f_{i+t}(0)} \tag{2.34}$$

for $i+t \geq j \geq i$.

Now letting $Q^c{}_{i,t}(z,w) = Q_{i+t}(z,w)$ one gets the block factorization step

$$Q_i(z,w) = a_{i+t}(z) a^*_{i+t}(w) [\sum_{j=0}^{t-1} k_{i+t-1-j} \frac{z^{j+1} - (w^*)^{j+1}}{z - w^*} (zw^*)^{t-1-j}] + (zw^*)^t Q_{i+t}(z,w) .$$

The generating function of the t columns of the corresponding triangular factor is $f_{i+t}(z) \frac{1 - (zw^*)^t}{1 - (zw^*)}$ while the corresponding block diagonal element has the generating function $[\sum_{j=0}^{t-1} k_{i+t-1-j} \frac{z^{j+1} - (w^*)^{j+1}}{z - w^*} (zw^*)^{t-1-j}]$.

A major application area for these results is to the problem of root distribution of polynomials, which we address next.

3: Linear system stability and root distribution of polynomials: A unified approach via inertia computation of Bezoutians

The problem of root distribution of polynomials with respect to (w.r.t.) a given curve in the complex plane is more than a hundred years old. The two most important curves in engineering application are the imaginary axis and the unit circle as they relate to the stability of continuous and discrete time linear time invariant systems respectively.

Our interest is in fast algorithms for solving these problems. There are many such solutions in the literature, especially the classical *Routh-Hurwitz* test (Routh [1877]) for the imaginary axis case and the *Schur-Cohn* test (Cohn [1924]) for the unit circle case. Later Marden [1949] and Jury [1964] independently developed a tabular form of the Schur-Cohn test. Recently some new results have been reported by Bistritz [1984] and by Lepschy, Mian and Viaro [1988]. Bistritz' test requires only half as many multiplications as for the Schur-Cohn or Marden-Jury tests. The test due to Lepschy et al. has no computational advantage over the well known Routh's test; it is just an alternative test. These various tests have all been derived using rather different methods and perspectives. These derivations do not provide any clues as to a unified framework, nor do they lead us to the derivation of equivalent or better new tests.

Moreover there are polynomials for which the Routh algorithm or the Schur-Cohn algorithm fail to compute the root distribution, leading to the so-called singularity problem; In fact all known algorithms for root distribution suffer from this problem and a thorough discussion of the singularity has been lacking for a long time.

Consider Routh's algorithm as an example. Since at every step one must perform at least one division by the leading element of the previous row (previous step), the procedure breaks down if this divisor becomes zero. This can happen in two ways: (i) if the given polynomial $f(s)$ and the associated polynomial $f(-s)$ (for real polynomials) share roots and (ii) if $f(s)$ has roots on both sides of the imaginary axis (not necessarily located symmetrically in pairs w.r.t the imaginary axis). However the converse of the second condition is not necessarily true. If all the elements of a row of Routh's array become zero, then condition (i) holds, while condition (ii) will hold if the leading element of a row is zero but the row itself does not vanish.

Routh was aware of these problems and he himself suggested modifications of his algorithm under these circumstances. Unfortunately Routh's modifications do not always work if we encounter a nonvanishing row with a zero leading element. Gantmakher [1959] suggested a different way of handling this problem, though it is

complicated and difficult to implement. Possibly for the first time Yeung [1983] came up with an efficient solution for polynomials with real coefficients; Yeung derived his solution from the point of view of completing generalized Sturm chains. Later Delsarte, Genin and Kamp [1985] independently derived the same solution as that of Yeung's using an index theory framework. Then Agashe [1985] came up with a complete solution for both real and the polynomials, using Rouche's theorem and the principle of argument.

A very similar scenario exists in the unit circle setting. As a matter of fact an efficient recursive solution for dealing with a singularity of the first kind has not yet been found. Originally Marden [1949] had suggested a procedure that requires starting fresh with a related polynomial of degree higher than the degree of the polynomial which poses the problem of singularity. This increase in degree could be up to t , when the degree of the polynomial that poses the problem of singularity is either $2t+2$ *or* $2t+1$. Another major drawback of this scheme is that it is not guaranteed that once a singularity is bypassed, another one will not be encountered in the process later. Yeung [1985] suggested another procedure along the lines of Marden's remedy. Yeung reduced the degree increment from t to unity for all values of t ; however, the second difficulty could not be removed. Then in 1985 Delsarte et. al. came up with a solution that could overcome the difficulties in the work of Marden [1949] and Yeung [1985]. However, their solution falls short of being efficient since it loses the recursive nature in overcoming singularities of the first kind. The basis of Marden's and Yeung's derivations was the principle of argument where as Delsarte et al. used their recently proposed index theory of pseudo-lossless functions.

It should be apparent by now that there are very many different derivations and interpretations in both the regular as well as the singular cases. A recent survey paper by Vaidyanathan and Mitra [1987] makes a unification attempt. However, despite several nice features, there are points in which it falls short of being a definitive treatment. For example, none of the known remedies for singular cases can be derived from the approach in their paper. Moreover, Vaidyanathan and Mitra [1987] did not address the issues regarding complex polynomials.

Recently we have been able to identify a unifying framework for the root distribution procedures (see Lev-Ari, Bistritz and Kailath [1987]) based on computing the inertia of certain generalized Bezout matrices constructed from the coefficients of the given polynomial, generalizing some classical results of Hermite [1856] (see also the fine survey paper of Krein and Naimark [1937]). The new point is that these Bezout matrices possess a generalized displacement structure in the sense that Kailath

and his associates have been studying for several years. They have shown (see e.g. Lev-Ari and Kailath [1986] also see the previous section) that displacement structure can be exploited to develop fast procedures ($O(N^2)$ as opposed to $O(N^3)$) for triangular factorization of such matrices. It turns out that application of these methods to the generalized Bezoutians readily yields all the classical tests and several equivalent ones in a simple and organized way. However the assumption of regularity (strong regularity of the rank profile) must be maintained. At this point it will be useful to begin to give a few more details.

A Bezout matrix B admits a generating function representation $B(z,w)$ given by,

$$B(z,w) = \frac{f(z)f^*(w) - f^\#(z)[f^\#(w)]^*}{d(z,w)} \tag{3.1}$$

where

$$B(z,w) = [1 \ z \ z^2 \ \ldots\ldots] \ B \ [1 \ w \ w^2 \ \ldots\ldots]^* \tag{3.2}$$

$$d(z,w) = [1 \ \ z] \begin{bmatrix} \alpha & \beta^* \\ \beta & \delta \end{bmatrix} [1 \ \ w]^* \tag{3.3}$$

and the sharp (#) denotes a suitably defined polynomial transformation. These matrices B serve to find the root distribution of the polynomial $f(z)$ w.r.t. the following partition of the complex plane,

$$\begin{aligned} \Omega_+ &= \{ z; d(z,z) > 0 \} \\ \Omega_0 &= \{ z; d(z,z) = 0 \} \\ \Omega_- &= \{ z; d(z,z) < 0 \} \end{aligned} \tag{3.4}$$

More specifically the inertia of the Bezout matrix B (i.e. the number of positive, zero and negative eigenvalues) indicates how many roots are shared by $f(z)$ and $f^\#(z)$ and how many remaining roots are in Ω_+ and in Ω_-. Ω_0 describes the curve w.r.t. which the root distribution is computed. In the imaginary axis case, $d(z,w) = z + w^*$ and $f^\#(z) = f^*(-z^*)$, while for the unit circle, $d(z,w) = 1 - zw^*$, $f^\#(z) = z^n f^*(1/z^*)$, n being the degree of $f(z)$. Generating functions of Bezout matrices are called Bezoutians.

$B(z,w)$ in (3.1) can be compactly represented as,

$$B(z,w) = \frac{G(z) \, J \, G^*(w)}{d(z,w)} \tag{3.5}$$

where $G(z)$ is an appropriate 1 x 2 row vector of polynomials and J is an appropriate $2x2$ non-singular Hermitian matrix. As noted earlier in section-2, this is in the so

called generalized displacement form of Lev-Ari and Kailath [1986], who showed that (see also the previous section of this proposal) fast triangular factorization of B (hence inertia calculation) via Sylvester's theorem can be obtained via the recursion (2.8). The recursion is started with $G_0(z) = G(z), J_0 = J$, and the updated Bezoutians at each step can be represented as

$$B_i(z,w) = \frac{G_i(z)J_iG_i^*(w)}{d(z,w)} .$$

In the above recursion the parameter ξ_i is arbitrary. The choice $\xi_i = 0$ yields the familiar triangular (LDL*) factorization of B (as mentioned in section-2), where as other choices yield non triangular factorizations with the $\{ B_i(\xi_i,\xi_i) \}$ for the non zero entries of the diagonal factor, which implies that the inertia of B is easily computed from the signs of the $\{ B_i(\xi_i,\xi_i) \}$.

It is clear that we have a great flexibility in choosing the parameters ξ_i,τ_i and the matrices U_i . Specific choices may help reduce computational requirements compared to the other cases.

For simplicity let us consider polynomials with real coefficients for now. Then the Bezoutian defined for imaginary axis problems is

$$B_I(z,w) = \frac{f(z)f^*(w) - f(-z)f^*(-w)]}{z + w^*} \tag{3.6}$$

Comparing (3.5) and (3.6) we find that

$$G(z) = [\, f(z) \;\; f^*(-z) \,] \,,\; d(z,w) = z+w^*$$

and,

$$J = J_T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

So the recursion (2.8) with the choice $\xi_i = 0$ becomes

$$(z-\xi_i)\, G_{i+1}(z) = G_i(z)\, \Theta_i(z) \tag{3.7a}$$

where

$$\Theta_i(z) = \left\{ I - \left\{ \frac{1}{z} + \frac{1}{\tau_i^*} \right\} J_i M_i \right\} U_i \tag{3.7b}$$

where

$$M_i = G_i^*(0) B_{I_i}^{-1}(0,0) G_i(0) \ . \tag{3.7c}$$

The choice $\tau_i = j\infty$ (since τ_i must be on the jw-axis) is an obvious one. Some

algebra yields

$$M_i = \begin{bmatrix} \rho_i & \rho_i \\ \rho_i & \rho_i \end{bmatrix} \qquad (3.7d)$$

where

$$\rho_i = \frac{f_i(0)}{2f_i'(0)} . \qquad (3.7e)$$

Therefore

$$\Theta_i(z) = \begin{bmatrix} 1 - \rho_i z^{-1} & \rho_i z^{-1} \\ -\rho_i z^{-1} & 1 + \rho_i z^{-1} \end{bmatrix} U_i , \qquad (3.7f)$$

where U_i can be any J unitary matrix. The simple choice J for U_i is convenient here because with this choice of U_i, if $G_i(z) = [\, f_i(z) \;\; f_i(-z)\,]$, then $G_i(z)\Theta(z)$ will have the form $[\, \alpha(z) \;\; \alpha(-z)\,]$, so that we can identify $G_{i+1}(z)$ as $[\, f_{i+1}(z) \;\; f_{i+1}(-z)\,]$. Then the recursions are

$$zf_{i+1}(z) = f_i(z) - \rho_i z^{-1}[\, f_i(z) - f_i(-z)\,] \qquad (3.8a)$$

$$zf_{i+1}(-z) = \rho_i z^{-1}[f_i(z) - f_i(-z)] - f_i(-z) \qquad (3.8b)$$

where

$$\rho_i = \frac{f_i(0)}{2f_i'(0)}$$

and the inertia of B_I can be computed from the signs of the $\{\rho_i\}$.

This apparently new looking recursion is (slightly) different in form, but completely equivalent in the amount of computation, to the well known Routh recursion, which utilizes the even and odd parts of a given polynomial. The Routh test can be obtained by adding and subtracting (3.8a,b). It can be directly obtained by starting with the so called immittance-type form of the Bezoutian.

At this point it will be useful to introduce the notion of scattering and immittance variables. The representation in (3.1) of $B(z,w)$ is called scattering-type, since

$$B \geq 0 \iff \sup_{z \in \Omega_+} \left| \frac{f^{\#}(z)}{f(z)} \right| \leq 1$$

This means that the ratio $f^{\#}(z)/f(z)$ can be interpreted as a (generalized) scattering function of a passive system. In particular this ratio is a continuous time scattering function $f^{*}(-z^{*})/f(z)$ of a lossless system when Ω_0= the jw axis, and a discrete time

scattering function $z^n f^*(1/z^*)/f(z)$ of a pseudo-lossless system when Ω_0= the unit circle.

Note that by a simple (invertible) transformation we can rewrite formula (3.1) as follows,

$$B(z,w) = \frac{a(z)b^*(w) + b(z)a^*(w)}{d(z,w)} \tag{3.9}$$

where

$$a(z) = \frac{1}{\sqrt{2}}[f(z) + f^\#(z)] \ , \text{ and}$$

$$b(z) = \frac{1}{\sqrt{2}}[f(z) - f^\#(z)] \ .$$

This representation is called immittance-type, since

$$B \geq 0 \quad \Longleftrightarrow \quad \inf_{z \in \Omega_+} \left\{ \mathrm{Re} \frac{b(z)}{a(z)} \right\} \geq 0$$

Thus the ratio $b(z)/a(z)$ is positive real in Ω_+ , which for $\Omega_o = jw$ *axis* (respectively the unit circle) corresponds to a continuous-time (respectively discrete time) impedance/admittance (= immittance) function. We have been able to identify the Routh's test with an immittance-type recursion while the Schur-Cohn test with a scattering type recursion for factorization of appropriate Bezout matrices.

A recursion is of "scattering type" when

$$G_i(z) = [\, f_i(z) \quad f_i^\#(z) \,] \quad \text{and} \quad J_i = J_T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} ,$$

since it would provide the scattering type representation for $B_i(z,w)$. Similarly a recursion is of "immittance-type" when

$$G_i(z) = [\, a_i(z)\, b_i(z) \,] \text{ and } J_i = J_I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ,$$

since this provides the immittance type representation for $B_i(z,w)$.

In the case, $f^\#(z) = f(-z)$ (considering real coefficients only), $a(z) = \sqrt{2}\, m(z)$ and $b(z) = \sqrt{2}\, n(z)$ where $m(z)$ and $n(z)$ are (using standard notations) respectively the even and odd parts of $f(z)$. So the immittance form of $B_I(z,w)$ is,

$$B_I(z,w) = \frac{2[m(z)n^*(w) + n(z)m^*(w)]}{z + w^*} \tag{3.10}$$

Since we are only interested in the inertia the factor of 2 could be ignored. The choice $\tau_i = j\infty$, and $U_i = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = J_I$ would yield for $\xi_i = 0$ the following recursion,

$$zm_{i+1}(z) = n_i(z) \tag{3.11}$$

$$zn_{i+1}(z) = m_i(z) - k_i z^{-1} n_i(z) \tag{3.12}$$

where

$$k_i = \lim_{z \to 0} zm_i(z)/n_i(z) = 2\rho_i = \frac{m_i(0)}{m_{i+1}(0)} \tag{3.13}$$

and inertia of B_I can be computed from the signs of $\{k_i\}$. This generates the familiar Routh recursion and the condition,

$\{k_i\}_{i=0}^{n-1} > 0 \iff$ signs of the leading $\iff$ stability.
elements of the Rouths's
array are all the same

Consider the following example. Let the given polynomial $f(z) = s^3 + 6s^2 + 11s + 6 = (s+1)(s+2)(s+3)$. Then the algorithm (3.11)-(3.12) leads to the following computations shown in array form

	s^0	s^2	K_i
$m_0(s)$	6	6	6/11
$m_1(s)$	11	1	121/60
$m_2(s)$	60/11		60/11
$m_3(s)$	1		

The above is the Routh array for the reverse polynomial $z^n f(1/z)$, which has the same root distribution w.r.t. the imaginary axis. Therefore the reversal of the positions of the coefficients is not material; in fact, this "reverse Routh array" is more convenient than the original array of Routh, since the starting polynomial in Routh's original method depends upon whether the highest power of $f(z)$ is even or odd, while our procedure removes any such ambiguity at once.

Since the above array could be formed only using the coefficients of $\{m_j(.)\}$, the even polynomials, a three term recursion is probably a more natural form for Routh's algorithm rather than the two term forms we have been describing. The desired three

term form could be found by eliminating $n_i(z)$ from (3.11)-(3.12),

$$z^2 m_{i+2}(z) = m_i(z) - \rho_i m_{i+1}(z) \tag{3.14}$$

and

$$m_0(z) = m(z) \text{ and } m_1(z) = n(z)/z \tag{3.15}$$

Since the 2-term scattering and immittance formulations both produced the Routh's array, so will the 3-term formulations. However the situation is not so degenerate if ξ_i is not on the jw axis. Keeping ξ_i real is necessary to keep the successive polynomials real. Then for a real $\xi_i \neq 0$, a scattering recursion would be

$$(z+\xi_i)\, f_{i+1}(z) = f_i(z) + k_i f_i(-z) \tag{3.16}$$

and

$$-\,(z-\xi_i)\, f_{i+1}(-z) = k_i f_i(z) + f_i(-z) \tag{3.17}$$

where

$$k_i = -f_i(-\xi_i)\,/f_i(\xi_i) \ .$$

It turns out that the inertia can be determined from the signs of the $\{\ \xi_i/(1-k_i^2)\ \}$. The two term recursion above produces the following backward decomposition in scattering variables

$$f_i(z) = \frac{1}{1-k_i^2}\ [\ (z+\xi_i)f_{i+1}(z) + k_i(z-\xi_i)f_{i+1}(-z)\ ] \tag{3.18}$$

This can be identified with the decomposition of Lepschy et. al. [1988] if we choose $\xi_i=1$. Since only the indices "i" and "$i+1$" are involved in the computation, Lepschy et al. in fact have a two-term scattering type recursion for the point of extraction $\xi_i = 1$. No one has presented any test that would correspond to a three term recursion in scattering variables for an arbitrary $\xi_i \neq 0$; The immittance-type recursions are also not yet associated with any known test. These gaps can readily be identified and filled with our approach, which sytematically classifies the vrious alternative tests (see Table 1).

Similar characterization of tests can be extended to the unit circle problems too. In fact the Schur-Cohn test corresponds to a two term scattering-type recursion and the Bistritz test [1984] corresponds to a three term immittance type recursion. These recursions of course use $\xi_i = 0$ as the point of extraction. A two term immittance-type recursion for $\xi_i = 0$ is not known to be related to any test as such. However such a test can be derived from the recursions in Bistritz, Lev-Ari and Kailath [1987]. The

Imaginary–axis

	Scattering	Immittance
Two term	Essentially Routh	Essentially Routh
Three term	Essentially Routh	Routh

Extraction point $\xi_i = 0$
(ξ_i on the boundary Ω_0)

Unit circle

	Scattering	Immittance
Two term	Schur–Cohn	Bistritz et al.[1987]
Three term	Szego [1939]	Bistritz [1984]

Extraction point $\xi_i = 0$
(ξ_i is not on the boundary Ω_0)

Imaginary–axis

	Scattering	Immittance
Two term	Lepschy et. al. [1988]	?
Three term	?	?

Extraction point $\xi_i = 1$
(ξ_i is not on the boundary Ω_0)

Unit circle

	Scattering	Immittance
Two term	?	Lev-Ari [1988]
Three term	?	Lev-Ari [1988]

Extraction point $\xi_i = 1$
(ξ_i on the boundary Ω_0)

Table 1 Classification of stability tests

three term scattering-type recursion can be related to the early work of Szegö [1939] on polynomials orthogonal on the unit circle.

Tests involving arbitrary (non-zero) extraction point in the unit circle context are not well known. Only recently Lev-Ari [1988] formulated new tests which utilize two-term and three-term immittance-type recursions for $\xi_i = 1$. In a way this test is analogous to the Routh scheme for the imaginary axis since both the procedures

The above discussion brings forth the following categorization of the tests in terms of our unified factorization framework. This can be well described by the tables shown in the following page. use an extraction point on the separating curve Ω_0 .

As far as the singular cases are concerned, very little is known the about factorization (inertia extraction) approach. Singular cases correspond to Bezout matrices with arbitrary rank profile (see Section 2.2). In fact,

$$B_{I_i}(0,0) = 0 \quad \Longleftrightarrow \quad \text{singularity in Routh recursion .}$$

If $B_{I_i}(0,0) = 0$, then the corresponding leading principal submatrix is singular and a triangular factorization can not be found without permutation of the matrix B_{I_i} . However a modified triangular factorization (as mentioned in section-2) of the form $B_I = LDL^*$, where L is lower triangular with unit diagonal element and D is a matrix with non-zero scalar and block entries only along the main diagonal can be computed. The imaginary axis Bezoutians are structurally close to the Quasi-Hankel matrices (a Bezout matrix is the product of a Quasi-Hankel matrix and a signature matrix (see Pal and Kailath [1988])) it has been possible to appropriately modify the results in Section 2.2 and derive immittance variable two term recursions for overcoming 'singularities of the first kind' arising in Routh's test. This derivation which is based on our factorization approach produces the same result as Yeung's [1983] test.

The sizes of these blocks in D are determined by the rank-profile of B_I and equals one more than twice the number of "shift-rows" in Yeung's [1983] array. So the inertia of B_I is computed from the inertia of D now. The block entries on the diagonal are also structured and the inertia of these blocks can be just computed by formula (see Pal and Kailath [1988]).

The recursion (3.11) - (3.12) gets modified as follows :

Case (i): $\lim_{z \to 0} m_i(z) \neq 0$, $\lim_{z \to 0} z^{-1} n_i(z) = 0$ and

t is the smallest integer such that $\lim_{z \to 0} z^{-(2t+1)} n_i(z) \neq 0$

$$zm_{i+1}(z) = n_i(z) \tag{3.19}$$

$$zn_{i+1}(z) = m_i(z) - k_i z^{-(2\iota+1)} n_i(z) \tag{3.20}$$

Case (ii): $\lim\limits_{z \to 0} m_i(z) = 0$

$$zm_{i+1}(z) = n_i(z) \tag{3.21}$$

$$zn_{i+1}(z) = m_i(z) \tag{3.22}$$

These modified recursions allow us to determine the triangular factors and hence inertia recursively, column by column rather than by block columns.

4. Fast Parallelizable Array Algorithms via Displacement Representations of Matrices

In this section we present a matrix formulation of the general factorization scheme described in Section 2. By applying this scheme to suitably chosen composite matrices, we obtain in Section 4.2 efficient solutions to several linear algebra problems.

Many signal processing problems require solving large systems of linear equations, either directly or via (weighted) least squares. The basic tools for the solutions are *triangular factorization* and *QR factorization*. These factorizations require $O(n^3)$ or $O(mn^2)$ flops (floating point operations) for an $m \times n$ matrix, which can often be excessively large. Therefore, attention focuses on *structured matrices,* with an eye both to computational reductions and to implementability in special purpose (parallel) hardware. Structured matrices arise in various problems of linear time invariant system. They include Toeplitz and close-to-Toeplitz matrices, Hankel and close-to-Hankel matrices, and Vandermonde and Krylov matrices. These matrices have certain shift-invariance properties as will be explained shortly, and are determined only by $O(\alpha m)$ parameters, where $\alpha \leq min(m,n)$. Therefore, one may expect that the solutions to the corresponding linear equations should be obtained with considerably less computation than for general matrices.

Our definition of structured matrices is that they are such that their appropriately defined *displacement.* has low rank. There are several useful definitions of displacement matrices, but we shall focus on two of them:

$$\nabla_{(F^f,F^b)}A = A - F^f A F^{bT}, \tag{4.1}$$

$$\Delta_{(F^f,F^b)}A = F^f A - A F^{bT}, \tag{4.2}$$

where F^f and F^b are chosen matrices. We shall call the matrices $\nabla_{(F^f,F^b)}$ and $\Delta_{(F^f,F^b)}$ the *Toeplitz and Hankel displacements* of A with respect to the *displacement operators* $\{F^f, F^b\}$, respectively. The displacement operators are chosen so that the displacements have ranks as low as possible. The ranks of the displacements $\nabla_{(F^f,F^b)}A$ and $\Delta_{(F^f,F^b)}A$ are called the *displacement ranks* of the matrix A. The computational complexity (as well as the space complexity) of fast algorithms is proportional to the displacement rank of the matrix. While our previous work focused mainly on the Toeplitz displacement (4.1), we are currently studying the application of the Hankel displacement (4.2) to factorization and inversion of close-to-Hankel and Vandermonde matrices. We should remark her that Heinig and Rost (1984) have already presented some efficient procedures for the inversion of such matrices. As

mentioned in the introduction their methodology and, consequently their computational procedures significantly differ from ours.

As an example of structured matrices, consider a regular Riccati differential equation,

$$\dot{P}(t) = FP(t) + P(t)F^T - P(t)H^T HP + GG^T, \tag{4.3}$$

where $F \in \mathbf{R}^{n \times n}$, $G \in \mathbf{R}^{n \times 1}$ and $H \in \mathbf{R}^{1 \times n}$. Then the stationary solution $\bar{P}$ of (4.3) has a low-rank Hankel displacement,

$$\Delta_{(F,F)}\bar{P} = GG^T - \bar{P}H^T H\bar{P}, \quad rank(\Delta_{(F,F)}\bar{P}) = 2.$$

Similarly, discrete-time Riccati equations have a rank 2 Toeplitz displacement. We first encountered these structured matrices in the study of Wiener-Hopf integral equations, which led us to the computationally efficient *Chandrasekhar filters* [see, e.g., Kailath (1972), Kailath (1973), Morf, Sidhu and Kailath (1974)].

Another important example is the well-known Toeplitz matrix. However, Toeplitz structure, though frequently encountered (stationary process, time-invariant systems, etc) and well exploited (Levinson and Schur algorithms), is not invariant under several operations arising in signal processing and numerical linear algebra - products, inversion, factorization, Schur complementations, etc. *What is invariant is the displacement structure.* We were able to re-derive both the Levinson and Schur algorithms using the low displacement rank property of Toeplitz matrix T. Notice that the displacement $\nabla_{(Z,Z)}T$, with respect to the $\{Z, Z\}$ has rank 2, where Z is the "shift" matrix,

$$\nabla_{(Z,Z)}T = \begin{bmatrix} t_0 & t_1 & \cdot & \cdot & t_n \\ t_1 & & & & \\ \cdot & & & & \\ t_n & & & & \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & & & \\ & & 1 & 0 \end{bmatrix}.$$

Similarly, for a Hankel matrix H

$$rank[\Delta_{(Z,Z)}H] = 2$$

and for a Krylov matrix K, i.e., the matrix whose ith row is $\mathbf{c}^T A^{i-1}$,

$$rank[\Delta_{(Z,A^{-1})}K] = 1$$

An important general property is that the family of matrices with a given displacement structure is closed under inversion and Schur-complementation (it is also closed under addition and in a slightly extended form, under multiplication).

Lemma 1. *For any nonsingular matrix A,*

$$rank\,[\nabla_{(F^f, F^b)} A\,] = rank\,[\nabla_{(F^{bT}, F^{fT})} A^{-1}],$$

$$rank\,[\Delta_{(F^f, F^b)} A\,] = rank\,[\Delta_{(F^{bT}, F^{fT})} A^{-1}].$$

Therefore, in particular,

$$rank\,[T - ZTZ^T] = rank\,[T^{-1} - Z^T T^{-1} Z] = 2$$

$$rank\,[ZH - HZ^T] = rank\,[Z^T H^{-1} - H^{-1} Z] = 2$$

Lemma 2. *Let the block matrix*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

have Hankel (Toeplitz) displacement rank α with respect to lower triangular displacement operators, $\{F^f, F^b\}$. Then the matrix,

$$\begin{bmatrix} O & O \\ O & S \end{bmatrix}, \quad S = D - CA^{-1}B$$

has the same Hankel (Toeplitz) displacement rank α with respect to $\{F^f, F^b\}$.

Lemma 1 and 2, whose proof we omit, suggest that it may be possible to compute inverses and Schur complements of structured matrices by operating on their *generators,* i.e., a matrix pair X, Y that satisfies the displacement equations

$$\nabla_{(F^f, F^b)} A = XY^T, \quad \text{or} \quad \Delta_{(F^f, F^b)} A = XY^T. \tag{4.4}$$

Doing so requires $O(\alpha m^2)$ computations, where α is the displacement rank of A, whereas working with the matrix A itself requires $O(m^3)$ computations.

We recall that successive Schur-complementation can be used to produce the triangular (LU) factorization of a matrix. We show how this fact can be used to obtain efficient algorithms for QR factorization, inversion, regularization, and solution of least squares problems. The key to obtaining these results is a combination of efficient algorithms for successive Schur-complementation applied to certain judiciously chosen "composite" (block) matrices.

We shall briefly outline the resulting so-called Generalized Schur algorithms.

4.1 Generalized Schur Algorithms.

To efficiently compute the Schur complement $D - CA^{-1}B$ of the matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

we need first to obtain a *proper generator* for M, i.e., a generator of the form,

$$X = \begin{bmatrix} * & 0 & \cdots & 0 \\ * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{bmatrix}, \quad Y = \begin{bmatrix} * & 0 & \cdots & 0 \\ * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{bmatrix}, \tag{4.5}$$

for the Toeplitz displacement, and

$$X = \begin{bmatrix} * & 0 & \cdots & 0 \\ * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & \cdots & 0 & * \\ * & \cdots & * & * \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & * \end{bmatrix}, \tag{4.6}$$

for the Hankel displacement. A non-proper generator of A can be converted to a proper one in several ways. One is applying the following matrices.

$$S_{i,j} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & c & & s_2 & \\ & & & \ddots & & \\ & & -s_1 & & c & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}, \quad c^2 + s_1 s_2 = 1. \tag{4.7}$$

Proper generators (4.5) or (4.6) can be obtained by using the matrix (4.7), or its special cases: Givens rotations, hyperbolic rotations and elementary matrices. By post-multiplying X and Y with a sequence of appropriate matrices $S_{i,j}$, we can transform X and Y to proper form with $O(\alpha m)$ computations.

The next step is to modify one column of X and Y. Repeating the same step (i.e., transformation to proper form followed by a modifications of a column) r times, where r is the size of the square block A in M produces the generator of the Schur complement $D - CA^{-1}B$. A detailed description of the algorithm is provided in the Appendix.

4.2 Applications of the Displacement Representation of Composite Matrices.

The above algorithms can be applied to the block matrices

$$A = \begin{bmatrix} T & I \\ I & O \end{bmatrix}, \begin{bmatrix} T^TT & I \\ I & O \end{bmatrix}, \begin{bmatrix} T_1 & T_2 \\ T_2^T & O \end{bmatrix}, \begin{bmatrix} T^TT & T^T \\ T & I \end{bmatrix}, \begin{bmatrix} T^TT & T^T \\ I & O \end{bmatrix},$$

to obtain *generalized Gohberg-Semencul formulas* [Kailath and Chun (1988)] or, equivalently, *displacement representations*, of the matrices,

$$T^{-1}, \quad (T^TT)^{-1}, \quad T_2^TT_1^{-1}T_2, \quad T(T^TT)^{-1}T^T, \quad (T^TT)^{-1}T^T.$$

Such composite matrices can be used to construct efficient numerical procedures for various linear algebra problems. In the sequel we describe several such applications, involving Toeplitz and close-to-Toeplitz matrices. Similar applications to Hankel matrices, upon which we shall not elaborate, also yield interesting results such as fast orthogonalization of Vandermonde matrices.

1. Orthogonalization.

Let $A \in \mathbf{R}^{m \times n}$ be a block-Toeplitz or a Toeplitz-block matrix, and let us define the block matrix,

$$M = \begin{bmatrix} -I & A & O \\ A^T & O & A^T \\ O & A & I \end{bmatrix}. \tag{4.8}$$

If we apply the generalized Schur algorithm to (4.8) after the mth step, we shall have a generator of

$$\begin{bmatrix} A^TA & A^T \\ A & I \end{bmatrix}$$

After another n steps of partial triangularization, we shall have

$$\begin{bmatrix} A^TA & A^T \\ A & I \end{bmatrix} = \begin{bmatrix} R^T \\ Q \end{bmatrix} \cdot [\, R \;\; Q^T \,] + \begin{bmatrix} O & O \\ O & S \end{bmatrix}.$$

Now, one can check that the matrices Q and R satisfy

$$A = QR, \qquad Q^TQ = I,$$

i.e., we obtained the QR factorization of A. This procedure will need $O(\alpha mn)$ flops.

If one wish to compute R^{-1} directly, then one can perform $m + n$ steps of partial triangularization with the matrix,

$$M = \begin{bmatrix} -I & A & O \\ A^T & O & I \\ O & I & O \end{bmatrix}$$

because

$$\begin{bmatrix} A^T A & I \\ I & O \end{bmatrix} = \begin{bmatrix} R^T \\ R^{-1} \end{bmatrix} \cdot [\, R \;\; R^{-T} \,] + \begin{bmatrix} O & O \\ O & S \end{bmatrix}.$$

2. Removing Forward Elimination of Square Systems

If one's primary interest in the factorization is to solve the square Toeplitz-block or block-Toeplitz system of equations,

$$A\,\mathbf{x} = \mathbf{b}, \tag{4.9}$$

then one might want to obtain the transformed right-side vector $\mathbf{y} = L^{-1}\mathbf{b}$, during the course of the factorization process. This can be done by performing the following partial triangular factorization of the matrix M,

$$M = \begin{bmatrix} A \\ -\mathbf{b}^T \end{bmatrix} = \begin{bmatrix} L \\ \mathbf{y}^T \end{bmatrix} \cdot L^T$$

whence the solution to (4.9) can be obtained by solving the triangular system of equations,

$$L^T \mathbf{x} = \mathbf{y}. \tag{4.10}$$

3. Removing Back-Substitution of Square Systems.

Even after eliminating the forward elimination step, from a hardware implementational point of view, the back-substitution step in (4.10) can still be quite cumbersome. This back-substitution process can also be eliminated by performing the partial factorization of the matrix,

$$M = \begin{bmatrix} A & -\mathbf{b} \\ I & 0 \end{bmatrix}.$$

Notice that the solution $\mathbf{x} = T^{-1}\mathbf{b}$ is the Schur complement of T in M. Therefore, after the n steps of partial triangularization, we shall have a generator of solution, and from the generator, we can read out the solution.

4. Solving Least Squares Problems without Back-substitutions.

To solve the weighted least squares problem, that minimizes

$$\|A_2(A_1\mathbf{x} - \mathbf{b})\|_2$$

where A_1 and A_2 are block-Toeplitz or Toeplitz-block, we form the following matrix.

$$M = \begin{bmatrix} -A_2 & A_1 & -\mathbf{b} \\ A_1^T & O & 0 \\ O & I & 0 \end{bmatrix}$$

Notice that the solution,

$$\mathbf{x} = (A_1^T A_2^{-1} A_1)^{-1} A_2^T \mathbf{b} \tag{4.11}$$

is the Schur complement of

$$\begin{bmatrix} -A_2 & A_1 \\ A_1^T & O \end{bmatrix}$$

Therefore, after the $m + n$ steps of generalized Schur algorithm, we shall have a generator of the solution (4.11), and the solution can be read out from the generator.

5. Regularization

If the given Toeplitz least squares system is particularly ill-conditioned, it is meaningless to compute the exact (least squares) solution, since small perturbation to the matrix can cause very large perturbations in the solution. In such cases, we solve the following regularized system

$$\begin{bmatrix} A \\ \eta I \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix}$$

by partial triangularization of

$$\begin{bmatrix} & O & A & b \\ & & \eta I & \mathbf{0} \\ A^T & \eta I & O & \mathbf{0} \\ & O & I & \mathbf{0} \end{bmatrix}$$

After the $m + 2n$ steps of generalized Schur algorithm, we shall have the solution.

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APPENDIX A

The following two theorems describe how to obtain a generator of a desired Schur complement as well as LU factorization.

Theorem 1. Fast LU factorization with Toeplitz generators.

Let

$$\nabla_{(F^f,F^b)}A^{(1)} = X^{(1)}Y^{(1)T}, \quad X^{(1)} = [\mathbf{x}_1^{(1)}, \mathbf{x}_2^{(1)}, \cdot \cdot, \mathbf{x}_\alpha^{(1)}], \quad Y^{(1)} = [\mathbf{y}_1^{(1)}, \mathbf{y}_2^{(1)}, \cdot \cdot, \mathbf{y}_\alpha^{(1)}],$$

where F^f and F^b are any strictly lower triangular, and $\{X^{(1)}, Y^{(1)}\}$ is proper. Then the first column of L and the first row of U, of $A^{(1)} = LU$ is given by

$$\mathbf{l}_1 = \mathbf{x}_1, \quad \mathbf{u}_1 = \mathbf{y}_1,$$

and the 1st order Schur complement $A^{(2)} = A^{(1)} - \mathbf{l}_1\mathbf{u}_1^T$ has a generator,

$$X^{(2)} = [F^f\mathbf{x}_1^{(1)}, \mathbf{x}_2^{(2)}, \cdot \cdot \cdot, \mathbf{x}_\alpha^{(1)}], \quad Y^{(2)} = [F^b\mathbf{y}_1^{(1)}, \mathbf{y}_2^{(1)}, \cdot \cdot \cdot, \mathbf{y}_\alpha^{(1)}] \tag{A.1}$$

i.e.,

$$\nabla_{(F^f,F^b)}A^{(2)} = X^{(2)}Y^{(2)T}.$$

Theorem 2. Fast LU factorization with Hankel generators.

Let

$$\Delta_{(F^f,F^b)}A^{(1)} = X^{(1)}Y^{(1)T}, \quad X^{(1)} = [\mathbf{x}_1^{(1)}, \mathbf{x}_2^{(1)}, \cdot \cdot, \mathbf{x}_\beta^{(1)}], \quad Y^{(1)} = [\mathbf{y}_1^{(1)}, \mathbf{y}_2^{(1)}, \cdot \cdot, \mathbf{y}_\beta^{(1)}],$$

where F^f and F^b are strictly lower triangular matrices, and $\{X^{(1)}, Y^{(1)}\}$ is proper. Then the first column of L and the first row of U, of $A^{(1)} = LU$ is given by

$$\mathbf{l}_1 = \mathbf{y}_\beta(1)(F^f)^{+}\mathbf{x}_\beta/\mathbf{l}_1(1), \quad \mathbf{u}_1 = -\mathbf{x}_1(1)(F^b)^{+}\mathbf{y}_1,$$

where the superscript + denotes the pseudo-inverse, and the 1st order Schur complement $A_2 = A_1 - \mathbf{l}_1\mathbf{u}_1^T$ has a generator,

$$X^{(2)} = [(\mathbf{x}_1^{(1)} - \mathbf{l}_1), \mathbf{x}_2^{(2)}, \cdot \cdot \cdot, \mathbf{x}_\beta^{(1)}], \quad Y^{(2)} = [\mathbf{y}_1^{(1)}, \mathbf{y}_2^{(1)}, \cdot \cdot \cdot, (\mathbf{y}_\beta^{(1)} - \mathbf{u}_1)] \tag{A.2}$$

i.e.,

$$\nabla_{(F^f,F^b)}A^{(2)} = X^{(2)}Y^{(2)T}.$$

The above theorems can be immediately translated into the following algorithms

Algorithm 1.

Input: A generator of $A^{(1)}$

Output: (1) A generator of the rth order Schur complement, (2) Triangular factorization.

for $i := 1$ **to** r **do begin**

Construct a proper generator of A_i

Obtain a generator of $A^{(i+1)}$ by (A.1);

end

Algorithm 2.

Input: A generator of $A^{(1)}$

Output: (1) A generator of the rth order Schur complement, (2) Triangular factorization.

for $i := 1$ **to** r **do begin**

Construct a proper generator of $A^{(i)}$

Obtain a generator of $A^{(i+1)}$ by (A.2);

end

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A. Bruckstein, T. Kailath, I. Koltracht and P. Lancaster, ''On The Reconstruction of Layered Media From Reflection Data,'' *SIAM J. Matrix Analysis.*

A. Dembo, ''Simple Proof of the Concavity of the Entropy Power with Respect to Added Gaussian Noise,'' *IEEE Trans. IT.*

A. Dembo, ''A New Proof of the Entropy Power Inequality,'' *IEEE Trans. IT.*

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J. Chun and T. Kailath, ''Systolic Array Implementation of the Square Root Chandrasekhar Filter,'' *IEEE Trans. on Automa. Control.*

7. Key Personnel/Bibliography/Advanced Degrees

SCIENTIFIC PERSONNEL SUPPORTED BY THIS PROJECT AND DEGREES AWARDED DURING 1986-1989:

Professor T. Kailath
Graduate Students: T. Citron, J. Chun, D. Pal, T. J. Shan
Research Associates: P. G. Gulak, H. Lev-Ari, R. Roy
Postdoctoral Fellows: Y. Bistritz, A. F. Bruckstein

Advanced Degrees

T. K. Citron, ''Algorithms and Architectures for Error Correcting Codes,'' Department of Electrical Enginering, Stanford University, August 1986.

T. J. Shan, ''Array Processing for Coherent Sources,'' Department of Electrical Enginering, Stanford University, March 1986.

R. H. Roy, ''Estimation of Signal Parameters Via Rotational Invariance Techniques,'' Department of Electrical Enginering, Stanford University, August 1987.

J-W. Chun, ''Fast Array Algorithms for Structured Matrices,'' Department of Electrical Enginering, Stanford University, June 1989.

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During 1961-1962 he worked at the Jet Propulsion Laboratories, Pasadena, CA, where he also taught part-time at the California Institute of Technology. From 1963-1968, he was Associate Professor and from 1968, Professor of Electrical Engineering at Stanford University. He was Director of the Information Systems Laboratory from 1971 through 1980, Associate Chairman of the Department of Electrical Engineering, from 1981 through 1987, and currently holds the Hitachi America Professorship in Engineering.

He has held shorter term appointments at several institutions around the world, including a Ford Fellowship in 1963 at UC Berkeley, a Guggenheim Fellowship in 1970 at the Indian Institute of Science, Bangalore, India, a Churchill Fellowship in 1977 at the Statistical Laboratory, Cambridge University, England, and a Michael Fellowship in 1984 at the Department of Theoretical Mathematics of The Weizmann Institute, Rehovot, Israel.

His research has involved a number of mathematical techniques and a variety of applications in statistical communications, control, information theory, linear systems and signal processing.

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